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Dependence upon Initial Conditions and Parameter of Solutions of Impulsive Differential Equations with Variable Structure

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A class of impulsive ordinary differential equations with variable structure is introduced. For the equations of this class it is characteristic that the changes of their right-hand sides and the impulses are realized at the moments when their solutions nullify switching functions (special functions with domains of definition coinciding with the phase space of the equations of the class). The initial value problem (with a parameter and without a parameter) for equations with variable structure and impulses is considered. Sufficient conditions for continuability and continuous dependence on the initial conditions and a parameter of their solutions are found. By means of the equations of this class the work of a hydraulic safety valve is simulated. The results obtained are used for the qualitative investigation of the model.

1. INTRODUCTION

The theory of differential equations with discontinuous right-hand side is comparatively well developed. This subject has been dealt with in numerous papers and some monographs (Philippov, 1985; Neimark, 1972). The wide use of various switches (relays) in automatic control systems has led to the development of almost all aspects of the qualitative theory of equations with discontinuous right-hand side. A number of important questions on this theory are considered in Andronov *et al.* (1959).

A straightforward generalization of these equations is to equations with variable structure. The investigation of equations with variable structure began with the work of Vogel (1951, 1953*a–d*). This theory was further developed in the investigations of, e.g., Myshkis and Hohryakov (1958) and

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Myshkis and Parshikova (1972). Many questions on the qualitative theory of equations with variable structure are studied in Emel'yanov (1970). For these equations it is characteristic that the changes of their right-hand sides are realized at the moments when their trajectories meet points of a set previously fixed and called the "critical set." The critical set is situated in the phase space of the equations and its topological structure may be quite diverse, but in most cases this consists of hypersurfaces also called "lines of switching," "lines of discontinuity," etc. The set of functions from which the right-hand sides of the equations are chosen usually consists of two functions. In Dishliev and Bainov (1987) the set from which the right-hand side of the equations is chosen is a one-parameter family of functions.

By means of impulsive differential equations, processes are studied which during their development are subject to perturbations causing abrupt changes of their state. The durations of the perturbations are negligible in comparison with the duration of the process, which is why they are assumed to take place "instantaneously" in the form of impulses. The mathematical theory of impulsive differential equations has developed comparatively recently. The first papers were by Mil'man and Myshkis (1960, 1963). In the last few years this theory has been developed intensively in relation to numerous applications to radio engineering, industrial robotics, biotechnology, etc. (Halany and Veksler, 1974; Matov, 1983; Pandit, 1980; Pandit and Deo, 1982; Rao and Rao, 1977; Simeonov and Bainov, 1983). The impulse effects for these equations take place at the moments when the integral curves meet points of a previously given set which we shall call the "impulse set." This set is situated in the extended phase space of the equations. Usually the impulse set consists of nonintersecting hypersurfaces. A particular case is that of differential equations with impulses at fixed moments. The mathematical theory of the latter equations is comparatively well studied.

The first paper in which impulsive differential equations with variable structure was investigated is Bainov and Milusheva (1981). In this paper the "critical set" and the "impulse set" coincide and the change of the structure and the impulses occur at the same moments (the moments when the integral curves of the equations of this class meet hypersurfaces of the critical set).

In the present paper a class of systems of impulsive ordinary differential equations with variable structure is introduced. For systems of this class it is characteristic that the changes of the structure (the changes of the right-hand sides of the systems) and the impulses occur at the moments when their solutions nullify special functions which we shall call switching functions, with domains of definition coinciding with the phase space of the systems of the class.

Before we formulate the problem which is the object of the present investigation, we shall consider an example from hydraulics, more precisely a model of a hydraulic safety valve (Matov, 1983). The main elements of the valve (Figure 1) are:

1. Intake manifold of volume V.
2. Conical shutoff of the valve with angle at the apex 2α and mass M_c.
3. Spiral spring with mass M_s and constant c_s.
4. Exhaust manifold.

Introduce the following notations: $x = x(t)$ is the displacement of the shutoff of the valve along the vertical line (Figure 1); P_1 and P_2 are the pressures, respectively, in the intake and exhaust manifolds; Q_1 and Q_2 are the incoming and outgoing flows of the fluid; D is the diameter of the bed of shutoff 2 (Figure 1). Henceforth, in order to simplify the model, the quantities Q_1 and P_2 will be assumed constant.

The shutoff of the valve is opened at the moment when the difference of the pressures in the intake and exhaust manifolds becomes greater than the force of the initial tension of the spring c_0 and is closed otherwise.

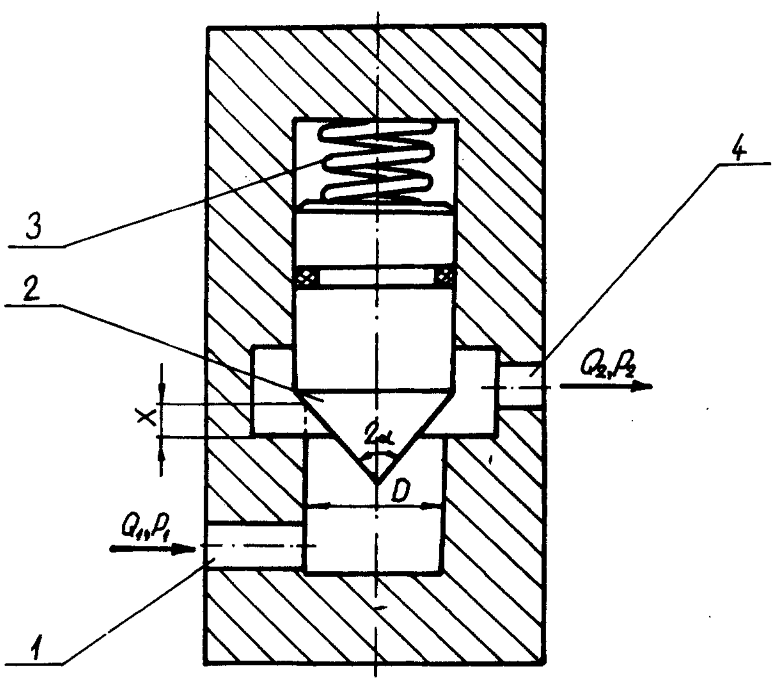

Fig. 1. Diagram of hydraulic safety valve.

The valve can be considered as a mechanical system which changes in the course of time. The state of this mechanical system is determined by the functions $x = x(t)$ and $P_1 = P_1(t)$, i.e., by the size of the opening of the valve and the pressure in the intake manifold.

The valve has two states.

First state. The valve is open, i.e., $x(t) > 0$. Then the displacement of the shutoff of the valve is described by the fundamental equation of dynamics $m\ddot{x} = P$, where $m = M_c + M_s/3$ and the force P is a vector sum of the initial tension of the spring c_0, the force of contraction of the spring $c_s x$, and the difference of the pressures P_1 and P_2. Hence, the function x satisfies the equation

$$\ddot{x} = \frac{1}{m}\left[\frac{\pi D^2}{4}(P_1 - P_2) - c_0 - c_s x\right], \qquad x > 0$$

By the laws of hydrodynamics, the change of the pressure is described by means of the differential equation

$$\dot{P}_1 = \frac{E}{V}(Q_1 - Q_2)$$

where E is the coefficient of the working fluid and the flow Q_2 going through the valve is given by the relation

$$Q_2 = \nu \pi D x \sin\alpha\, [2(P_1 - P_2)/\omega]^{1/2}$$

In the last equality ν is the coefficient of the flow Q_2, ω is the density of the fluid, and $\pi D x \sin\alpha$ is the approximate area of the opening formed when the shutoff of the valve is displaced at distance x. Denote by y the velocity of motion of the shutoff, i.e., $y = \dot{x}$, and by w denote the set of parameters participating in the differential equations describing the state of the valve, i.e., $w = (D, M_c, M_s, c_0, c_s, \alpha, \nu, \omega)$. Then the normalized system simulating this state of the mechanical system becomes

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \frac{1}{m} P(x, P_1, w) \\ \dot{P}_1 &= \frac{E}{V}(Q_1 - Q_2(x, P_1, w)) \end{aligned} \tag{1}$$

Second state. The valve is closed, i.e., $x(t) = 0$. In this case the velocity y and the flow Q_2 are equal to zero. The state of the mechanical system is simulated by means of the system of differential equations

$$\begin{aligned} \dot{x} &= 0 \\ \dot{y} &= 0 \\ \dot{P}_1 &= \frac{E}{V} Q_1 \end{aligned} \tag{2}$$

The transition of the valve from the first state to the second takes place at the moments $\tau_1, \tau_2, \ldots$ when the shutoff of the valve takes the lowermost position, i.e.,

$$x(\tau_i) = 0, \qquad i = 1, 2, \ldots$$

Denote by $\varphi = \varphi(P_1)$ the function

$$\varphi(P_1) = \pi D^2(P_1 - P_2)/4 - c_0 = P(x, P_1, w)|_{x=0}$$

When the function φ is negative, the valve is in the second state. The transition of the mechanical system from the second state to the first takes place at the moments $\theta_1, \theta_2, \ldots$ when the function φ vanishes, i.e., the following equalities hold:

$$\varphi(P_1(\theta_i)) = 0, \qquad i = 1, 2, \ldots$$

At the moments τ_i and θ_i the system of differential equations describing the state of the valve changes its right-hand side. More precisely, at the moments τ_i, system (1) is replaced by system (2), and at the moments θ_i, (2) is replaced by (1). This means that the system describing the state of the valve has variable structure. At the moments θ_i the solution of this system is continuous, i.e., at the moments θ_i the system is not subject to an impulse effect. By the transition from the first state to the second the shutoff of the valve takes the lowermost position and its velocity vanishes, independent of its magnitude. Hence, at the moments τ_i the system of differential equations simulating the action of the valve is subject to an impulse effect. Mathematically, this is expressed by

$$\Delta y(t)|_{t=\tau_i} = y(\tau_i + O) - y(\tau_i) = -y(\tau_i), \qquad i = 1, 2, \ldots \tag{3}$$

The solution of the simulating system is a piecewise continuous function with points of discontinuity of the first kind, namely the points $\tau_1, \tau_2, \ldots$ at which the impulses are realized.

Systems similar to the one considered are also obtained by the mathematical simulation of many processes in robotics, impulse technology, biotechnology, etc.

One of the goals of the present paper is to introduce a sufficiently general mathematical model comprising both systems describing the work of the safety valve and similar mathematical models. For these systems further fundamental, qualitative, and asymptotic theories should be developed and all investigations traditional for ordinary differential equations should be carried out. In this paper the respective initial value problem is considered and sufficient conditions for a continuous dependence of its solution on the initial conditions and a parameter are found.

2. STATEMENT OF THE PROBLEM

We call a system of ordinary differential equations with variable structure and impulses (SODEVSI) the following set of objects and relations among them:

1. The set of functions $\alpha_i: D \to R$, $i = 1, 2, \ldots$, where D is a domain in R^n. These functions we shall call switching functions.
2. Systems of differential equations with a parameter

$$\partial x/\partial t = f_i(t, x, \mu), \qquad i = 1, 2, \ldots$$

where

$$f_i: R^+ \times D \times M \to R^n, R^+ = [0, +\infty), M = [\mu_1, \mu_2], \mu_1, \mu_2 \in R$$

3. Equalities

$$\Delta x(t, \mu)|_{t=\tau_i} = x(\tau_i + 0, \mu) - x(\tau_i, \mu) = I_i(x(\tau_i, \mu), \mu), \qquad i = 1, 2, \ldots$$

where $I_i: D \times M \to R^n$ and $\tau_1, \tau_2, \ldots$ are the moments when the changes of the structure and the impulses occur. By means of the above equalities, the magnitudes and the directions of the impulse perturbations are determined.

We shall describe the way in which the solution $x(t, \mu)$ of a SODEVSI with initial condition

$$x(\tau_0, \mu) = x_0(\mu)$$

is found, where $\tau_0 \in R^+$ and $x_0: M \to D$. For $\tau_0 \leq t < \tau_1$ the solution of the SODEVSI coincides with the solution $x_1(t, \mu)$ of the problem

$$\partial x/\partial t = f_1(t, x, \mu), \qquad x_1(\tau_0, \mu) = x_0(\mu)$$

Here τ_1 is a constant or ∞ and is defined by means of the equality

$$\tau_1 = \begin{cases} \infty & \text{if} \quad \alpha_1(x_1(t, \mu)) \neq 0, \qquad t > \tau_0 \\ \inf\{t;\ t > \tau_0, \alpha_1(x_1(t, \mu)) = 0\} & \text{otherwise} \end{cases}$$

If $\tau_1 < \infty$, we assume that at the moment τ_1 the solution of the SODEVSI is equal to $x_1(\tau_1, \mu)$. At this moment it is subject to an impulse effect which is expressed by means of the equality

$$x(\tau_1+0, \mu) = x_1(\tau_1, \mu) + I_1(x_1(\tau_1, \mu))$$

If $\tau_1 = \infty$, then a change of the structure and impulse effect do not occur. Let $\tau_1 < \infty$. Then for $\tau_1 < t < \tau_2$ the solution of the SODEVSI coincides with the solution $x_2(t, \mu)$ of the problem

$$\partial x/\partial t = f_2(t, x, \mu), \qquad x_2(\tau_1, \mu) = x_1(\tau_1, \mu) + I_1(x_1(\tau_1, \mu), \mu)$$

The following holds:

$$\tau_2 = \begin{cases} \infty & \text{if} \quad \alpha_2(x_2(t, \mu)) \neq 0, \qquad t > \tau_1 \\ \inf\{t;\, t > \tau_1,\, \alpha_2(x_2(t, \mu)) = 0\} & \text{otherwise} \end{cases}$$

If $\tau_2 < \infty$, we assume that $x(\tau_2, \mu) = x_2(\tau_2, \mu)$. After an impulse the solution of the SODEVSI takes the value

$$x(\tau_2+0, \mu) = x_2(\tau_2, \mu) + I_2(x_2(\tau_2, \mu), \mu)$$

If $\tau_2 = \infty$, then changes of the structure and impulses do not occur for $t > \tau_1$. If $\tau_2 < \infty$, then for $\tau_2 < t < \tau_3$ the solution is determined by the problem

$$\partial x/\partial t = f_3(t, x, \mu), \qquad x_3(\tau_2, \mu) = x_2(\tau_2, \mu) + I_2(x_2(\tau_2; \mu), \mu)$$

etc. The solution of the SODEVSI is a piecewise continuous function with points of discontinuity $\tau_1, \tau_2, \ldots$ at which it has a discontinuity of the first kind. The solution is continuous from the left at the impulse moments.

We shall denote briefly the initial value problem considered for SODEVSI as follows:

$$\partial x/\partial t = f_i(t, x, \mu), \qquad \tau_{i-1} < t \leq \tau_i \tag{4}$$

$$\Delta x(t, \mu)|_{t=\tau_i} = I_i(x(\tau_i, \mu), \mu), \qquad i = 1, 2, \ldots \tag{5}$$

$$x(\tau_0, \mu) = x_0(\mu) \tag{6}$$

3. AUXILIARY RESULTS

Introduce the following notations: by $\langle \cdot, \cdot \rangle$ denote the scalar product in R^n; by $\|\cdot\|$ the Euclidean norm in R^n; by $\rho(A, B)$ the distance between the nonempty sets $A, B \subset R^n$ induced by the Euclidean norm, i.e., $\rho(A, B) = \inf\{\|a - b\|;\, a \in A,\, b \in B\}$; by $x(t; \tau_0^*, x_0^*, \mu^*)$ denote the solution of system (4), (5) with initial condition $x(\tau_0^*, \mu^*) = x_0^*$; $x_i = x(\tau_i; \tau_0, x_0(\mu), \mu)$; $x_i^+ = x_i + I_i(x_i, \mu)$; $A_i = \{x;\, \alpha_i(x) = 0,\, x \in D\}$, $i = 1, 2, \ldots$.

Lemma 1. Let the function $\varphi:[\tau_0, T]\to R$, $\tau_0, T\in R$, satisfy the following conditions:

1. $\varphi(\tau_1)=\varphi(\tau_2)$, where $\tau_0\le\tau_1<\tau_2\le T$.

2. The function $d\varphi/dt$ is Lipschitz continuous in $[\tau_0, T]$ with constant $L>0$.

Then $\tau_2-\tau_1>|d\varphi(\tau)/dt|/L$ for $\tau_1\le\tau\le\tau_2$.

Proof. From Condition 1 it follows that there exists a point τ', $\tau_1<\tau'<\tau_2$, such that $d\varphi(\tau')/dt=0$. But then

$$|d\varphi(\tau)/dt|=|d\varphi(\tau)/dt-d\varphi(\tau')/dt|\le L|\tau-\tau'|<L(\tau_2-\tau_1)$$

Thus, Lemma 1 is proved. ∎

Let the following conditions denoted by (A) be fulfilled for any $i=1,2,\ldots$.

A1. The functions f_i are Lipschitz continuous in $R^+\times D$ with constant $F>0$ independent of μ.

A2. There exist constants $M_i>0$ such that $\|f_i(t,x,\mu)\|\le M_i$ for $(t,x,\mu)\in R^+\times D\times M$.

A3. For any point $(\tau_0^*, x_0^*, \mu^*)\in R^+\times D\times M$ the solution of the problem

$$dx/dt=f_i(t,x,\mu^*),\qquad x(\tau_0^*)=x_0^*$$

does not leave the set D for $t\ge\tau_0^*$.

A4. The functions $\partial\alpha_i/\partial x=(\partial\alpha_i/\partial x_1,\ldots,\partial\alpha_i/\partial x_n)$ are Lipschitz continuous in D respectively with constants $L_i\ge 0$, $x=(x_1,\ldots,x_n)$.

A5. $\|\partial\alpha_i(x)/\partial x\|\le K_i$, $x\in D$, $K_i\ge 0$.

A6. There exist constants $\rho_i\ge 0$ for which one of the following two conditions hold:

A6.1. (i) $(I+I_i)$: $A_i\times M\to D$, where I is the identity in $R^n\times M$, and (ii) $|\alpha_i(x+I_{i-1}(x,\mu))|\ge\rho_i$, $(x,\mu)\in A_{i-1}\times M$, where $I_0(x)=0$, $A_0=D$.

A6.2. (i) $(I+I_{i-1})$: $A_{i-1}\times M\to A_i$, and (ii) there exists a point τ, $\tau_{i-1}<\tau\le\tau_i$, such that

$$|\langle\partial\alpha_i(x(\tau;\tau_0,x_0(\mu),\mu))/\partial x,\; f_i(\tau,x(\tau;\tau_0,x_0(\mu),\mu),\mu)\rangle|\ge\rho_i,\qquad \mu\in M$$

A7. The series $\sum_1^\infty \rho_i/[M_i^2L_i+K_iF(1+M_i)]$ is divergent.

We shall show that if conditions (A) hold, then the impulse moments $\tau_1,\tau_2,\ldots$ have no density point. More precisely, the following theorem is valid.

Theorem 1. Let conditions (A) hold. Then:

(i) The solution of problem (4)-(6) exists and is unique in the interval $[\tau_0,\infty)$, $\tau_0\ge 0$.

(ii) Each finite interval $[\tau_0, T]$ contains a finite number of impulse moments $\tau_1, \tau_2, \ldots$.

Proof. Set $x(t) = x(t; \tau_0, x_0(\mu), \mu)$. From conditions A1, A3, and A6.1i (A6.2i) it follows that the solution of problem (4)-(6) exists and is unique in each of the intervals $[\tau_0, \tau_1]$, $(\tau_1, \tau_2]$, $(\tau_2, \tau_3]$, ... If $x(t)$ nullifies a finite number of the switching functions α_i, then the solution of problem (4)-(6) is defined for all $t \geq \tau_0$ and in the interval $[\tau_0, T]$ it has a finite number of impulse moments.

Let $x(t)$ nullify each of the functions α_i. We shall evaluate the difference $\tau_i - \tau_{i-1}$, $i = 1, 2, \ldots$, if condition A6.1 holds. We have

$$\begin{aligned}\rho_i &\leq |\alpha_i(x_{i-1}^+)| \\ &= |\alpha_i(x_i) - \alpha_i(x_{i-1}^+)| \\ &= |\partial\alpha_i(x(\tau))/\partial x_1 \cdot x_1'(\tau_0) + \cdots + \partial\alpha_i(x(\tau))/\partial x_n\, x_n'(\tau)|(\tau_i - \tau_{i-1}) \\ &= |\langle \partial\alpha_i(x(\tau))/\partial x, f_i(\tau, x(\tau), \mu)\rangle|(\tau_i - \tau_{i-1}), \qquad \tau_{i-1} < \tau < \tau_i \end{aligned} \tag{7}$$

From inequality (7) and conditions A2 and A5 we obtain the estimate

$$\tau_i - \tau_{i-1} \geq \rho_i / K_i M_i, \qquad i = 1, 2, \ldots \tag{8}$$

We shall evaluate the difference $\tau_i - \tau_{i-1}$, $i = 1, 2, \ldots$, if condition A6.2 holds. Taking into account $x_{i-1} \in A_{i-1}$ and condition A6.2i, we obtain that $x_{i-1}^+ = x_{i-1} + I_{i-1}(x_{i-1}, \mu) \in A_i$, i.e., $\alpha_i(x_{i-1}^+) = 0$. Consider the function

$$\varphi(t) = \begin{cases} \alpha_i(x_{i-1}^+), & t = \tau_{i-1} \\ \alpha_i(x(t)), & \tau_{i-1} < t \leq \tau_i \end{cases}$$

the following equalities hold:

$$\varphi(\tau_{i-1}) = \varphi(\tau_i) = 0 \tag{9}$$

We shall show that the function

$$d\varphi(t)/dt = \langle \partial\alpha_i(x(t))/\partial x, f_i(t, x(t), \mu)\rangle$$

is Lipschitz continuous for $\tau_{i-1} < t \leq \tau_i$. In fact, let the points $t_1, t_2 \in (\tau_{i-1}, \tau_i]$. Then

$$\begin{aligned}&|d\varphi(t_1)/dt - d\varphi(t_2)/dt| \\ &\quad \leq |\langle \partial\alpha_i(x(t_1))/\partial x, f_i(t_1, x(t_1), \mu)\rangle \\ &\qquad - \langle \partial\alpha_i(x(t_2))/\partial x, f_i(t_1, x(t_1), \mu)\rangle| \\ &\qquad + |\langle \partial\alpha_i(x(t_2))/\partial x, f_i(t_1, x(t_1), \mu)\rangle \\ &\qquad - \langle \partial\alpha_i(x(t_2))/\partial x, f_i(t_2, x(t_2), \mu)\rangle| \\ &\quad \leq M_i L_i \|x(t_1) - x(t_2)\| + K_i F(|t_1 - t_2| + \|x(t_1) - x(t_2)\|) \\ &\quad = (M_i L_i M_i + K_i F(1 + M_i))|t_1 - t_2| \end{aligned} \tag{10}$$

In view of (9), (10), and Lemma 1 we obtain the estimate

$$\tau_1 - \tau_{i-1} \geq \frac{|d\varphi(\tau)/d\tau|}{M_i^2 L_i + K_i F(1+M_i)}, \qquad \tau_{i-1} \leq \tau \leq \tau_i$$

From the last inequality and condition A6.2ii we find that

$$\tau_i - \tau_{i-1} \geqq \frac{\rho_i}{M_i^2 L_i + K_i F(1+M_i)} = u_i \tag{11}$$

The following inequalities hold:

$$\frac{\rho_i}{K_i M_i} \geq \begin{cases} u_i, & F \geq 1 \\ Fu_i, & 0 < F < 1, \quad i = 1, 2, \ldots \end{cases} \tag{12}$$

Let $0 < F < 1$ (the case $F \geq 1$ is considered analogously). By condition A7, the series $\sum_1^\infty Fu_i$ is divergent. Hence, there exists a number k such that

$$F(u_0 + u_1 + \cdots + u_k) > T - \tau_0 \tag{13}$$

From inequalities (8) and (11)-(13) we obtain that

$$(\tau_1 - \tau_0) + (\tau_2 - \tau_1) + \cdots + (\tau_{k+1} - \tau_k) > T - \tau_0 \tag{14}$$

i.e., $\tau_{k+1} > T$, which means that for $\tau_0 < t < T$ a finite number of impulses is realized.

Similarly to inequality (14), it can be shown that for an arbitrary interval $[\tau_0, \tau]$ there exists a number $k = k(\tau)$ such that

$$[\tau_0, \tau] \subset [\tau_0, \tau_1] \cup \left(\bigcup_1^k (\tau_i, \tau_{i+1}]\right)$$

Since the solution $x(t)$ is defined in each of the intervals $(\tau_{i-1}, \tau_i]$, $i = 1, 2, \ldots$, it follows that $x(t)$ is defined in the interval $[\tau_0, \tau]$, which implies that the solution of problem (4)-(6) exists and is unique for $\tau_0 \leq t < +\infty$.

This completes the proof of Theorem 1. ■

Introduce the notation $x^*(t) = x(t; \tau_0, x_0(\mu^*), \mu^*)$, i.e., $x^*(t)$ is the solution of the problem

$$\partial x/\partial t = f_i(t, x, \mu^*), \qquad \tau_{i-1}^* < t \leq \tau_i^*, \qquad \tau_0^* = \tau_0 \tag{15}$$

$$\Delta x(t, \mu^*)|_{t=\tau_i^*} = I_i(x(\tau_i^*, \mu^*), \mu^*), \qquad i = 1, 2, \ldots \tag{16}$$

$$x(\tau_0, \mu^*) = x_0(\mu^*) \tag{17}$$

where the parameter $\mu^* \in M$ and the moments $\tau_1^*, \tau_2^*, \ldots$ are determined analogously to $\tau_1, \tau_2, \ldots$.

Denote by (B) the following condition:

B. The function x_0 is continuous in M.

Lemma 2. Let conditions (A) and (B) hold and for $\tau_0 \le t \le T$ the solution of problem (4)-(6) does not nullify the function α_1.

Then, if $\mu^* \in M$ and the number $|\mu^* - \mu|$ is small enough, then the solution of problem (4)-(6) also does not nullify the function α_1 for $\tau_0 \le t \le T$.

Proof. Consider the sets

$$A_1 = \{x;\ \alpha_1(x) = 0,\ x \in D\}$$
$$B = \{x;\ x = x(t),\ \tau_0 \le t \le T\}$$

and

$$C = \{x;\ x = x^*(t),\ \tau_0 \le t \le T\}$$

The sets A_1 and B are closed, B is bounded, and $A_1 \cap B = \varnothing$. Hence,

$$\rho(A_1, B) = \eta > 0 \tag{18}$$

In view of condition (B) and the theorem of the continuous dependence of the solution of a system of differential equations (without variable structure and impulses) on the initial condition and a parameter (henceforth, for the sake of brevity, this theorem will be called the theorem of the continuous dependence), it follows that for a sufficiently small value of $|\mu^* - \mu|$, $\mu^* \in M$, the following inequality holds:

$$\|x(t) - x^*(t)\| < \eta, \qquad \tau_0 \le t \le T$$

i.e.,

$$g(B, C) < \eta \tag{19}$$

From (18) and (19) we obtain the estimate

$$\rho(A_1, C) \ge \rho(A_1, B) - \rho(B, C) > 0$$

which means that $A_1 \cap C = \varnothing$. Hence, the solution $x^*(t)$ for $\tau_0 \le t \le T$ does not nullify the function α_1.

Thus, Lemma 2 is proved. ■

We shall say that condition (C) is satisfied if the following condition holds.

C. There exist positive constants δ_i such that for any $\tau' \in (\tau_i - \delta_i, \tau_i)$, $\tau'' \in (\tau_i, \tau_i + \delta_i)$, and $\mu \in M$ the following inequalities hold:

$$\alpha_i(x_i(\tau', \mu))\langle \partial\alpha_i(x_i(\tau'', \mu))/\partial x,$$
$$f_i(\tau'', x_i(\tau'', \mu), \mu)\rangle < 0, \qquad i = 1, 2, \ldots$$

where by $x_i(t, \mu)$ we have denoted respectively the solutions of the problems

$$\partial x/\partial t = f_i(t, x, \mu), \qquad x_i(\tau_{i-1}, \mu) = x_{i-1}(\tau_{i-1}, \mu) + I_{i-1}(x_{i-1}(\tau_{i-1}, \mu), \mu)$$

Lemma 3. Let conditions (A)-(C) hold. For $\tau_0 < t < T$ let the solution of problem (4)-(6) nullify the function α_1.

Then, if $\mu^* \in M$ and the number $|\mu^* - \mu|$ is small enough, the solution of problem (15)-(17) also nullifies the function α_1 for $\tau_0 < t < T$.

Proof. Without loss of generality we can assume that the constant δ_1 [from condition (C)] satisfies the inequality $\delta_1 < \tau_1 - \tau_0$. The function $\alpha_1(x_1(t, \mu))$ does not change its sign for $\tau_0 < t < \tau_1$; thus, assume that, for the sake of definiteness, the following inequality holds:

$$\alpha_1(x_1(\tau', \mu)) > 0, \qquad \tau_1 - \delta_1 < \tau' < \tau_1, \qquad \mu \in M \tag{20}$$

[the case $\alpha_1(x_1(\tau', \mu)) < 0$ is considered analogously]. Then

$$\alpha_1(x_1(\tau_1, \mu)) = 0 \tag{21}$$

$$\langle \partial\alpha_1(x_1(\tau'', \mu))/\partial x, f_1(\tau'', x_1(\tau'', \mu), \mu)\rangle < 0, \qquad \tau_1 < \tau'' < \tau_1 + \delta_1 \tag{22}$$

Assume that for $\tau_1 < t < \tau_1 + \delta_1$ the inequality $0 \leq \alpha_1(x_1(t, \mu))$ holds. From (21) it follows that

$$\begin{aligned} 0 &\leq \alpha_1(x_1(t, \mu)) - \alpha_1(x_1(\tau_1, \mu)) \\ &= \langle \partial\alpha_1(x_1(\tau''', \mu))/\partial x, f_1(\tau''', x_1(\tau''', \mu), \mu)\rangle (t - \tau_1) \\ &\qquad \tau_1 < \tau''' < t < \tau_1 + \delta_1 \end{aligned}$$

The last inequality contradicts (22). Hence, there exists a point τ^{iv}, $\tau_1 < \tau^{iv} < T$, such that

$$\alpha_1(x_1(\tau^{iv}, \mu)) < 0 \tag{23}$$

Suppose that for each $\mu^* \in M$ the solution $x^*(t)$ of problem (15)-(17) does not nullify the function α_1 for $\tau_0 \leq t$. Then, in view of (20), (23), and the theorem of continuous dependence, we establish that for a sufficiently small value of $|\mu^* - \mu|$, $\mu^* \in M$, the following inequalities hold:

$$\alpha_1(x^*(\tau')) > 0, \qquad \alpha_1(x^*(\tau^{iv})) < 0$$

From the last inequalities and the fact that the function $\varphi(t) = \alpha_1(x^*(t))$ is continuous for $t > \tau_0$ we conclude that there exists a point τ^v, $\tau^i < \tau^v < \tau^{iv}$, such that

$$\alpha_1(x^*(\tau^v)) = 0$$

which contradicts the assumption.

This completes the proof of Lemma 3. ■

4. MAIN RESULTS

Definition 1. We shall say that the solution of problem (4)-(6) depends continuously on the parameter $\mu \in M$ if

$$(\forall \varepsilon, \eta > 0)(\exists \delta\colon \delta(\varepsilon, \eta) > 0)(\forall \mu^* \in M, |\mu^* - \mu| < \delta)(\forall \tau_0 \in R^+) \Rightarrow$$
$$\|x(t; \tau_0, x_0(\mu), \mu) - x(t; \tau_0, x_0(\mu^*), \mu^*)\| < \varepsilon$$
$$t \geq \tau_0, \qquad |t - \tau_i| > \eta, \qquad i = 1, 2, \ldots$$

Denote by (D) the following condition.

D. The functions I_i, $i = 1, 2, \ldots$, are continuous in $D \times M$.

Theorem 2. Let conditions (A)-(D) hold. Let $\tau_0 < \tau_1 < T < \tau_2$. Then the solution of problem (4)-(6) depends continuously on the parameter $\mu \in M$ for $\tau_0 \leq t \leq T$.

Proof. Let ε and η be arbitrary positive constants. Without loss of generality we assume that $\eta < \min(\tau_1 - \tau_0, T - \tau_1)$. For $\tau_0 \leq t \leq \tau_1 - \eta$ the solution $x(t)$ does not nullify the function α_1. Then by Lemma 2 there exists a constant $\delta' > 0$ such that if $\mu^* \in M$ and $|\mu^* - \mu| < \delta'$, then the solution $x^*(t)$ also does not nullify the function α_1 for $\tau_0 \leq t \leq \tau_1 - \eta$. For $\tau_0 \leq t \leq \tau_1 + \eta$ the solution $x(t)$ nullifies the function α_1. From Lemma 3 it follows that there exists $\delta'' > 0$ such that if $\mu^* \in M$ and $|\mu^* - \mu| < \delta''$, then the solution $x^*(t)$ also nullifies the function α_1 for $\tau_0 < t < \tau_1 + \eta$. Hence, if $|\mu^* - \mu| < \min(\delta', \delta'')$, then $|\tau_1^* - \tau_1| < \eta$. The last inequality also implies that

$$\lim_{\mu^* \to \mu (\mu^* \in M)} \tau_1^* = \tau_1 \tag{24}$$

We shall evaluate $\|x^*(\tau_1^*) - x(\tau_1)\|$. Assume that $\tau_1 \leq \tau_1^*$. The case $\tau_1 > \tau_1^*$ is considered analogously. From the theorem of continuous dependence it follows that there exists a constant $\delta''' > 0$ such that if $\mu^* \in M$ and $|\mu^* - \mu| < \delta'''$, then

$$\|x^*(t) - x(t)\| < \varepsilon, \qquad \tau_0 \leq t \leq \tau_1$$

Since

$$x^*(t) = x^*(\tau_1) + \int_{\tau_1}^{t} f_1(\tau, x^*(\tau), \mu)\, d\tau, \qquad \tau_1 \leq t \leq \tau_1^* \tag{25}$$

then in view of condition A2 we obtain the estimate

$$\|x^*(\tau_1^*) - x^*(\tau_1)\| \leq M_1(\tau_1^* - \tau_1) < M_1 \eta$$

From the last inequality and (25) for $|\mu^*-\mu|<\min(\delta', \delta'', \delta''')$ we deduce the estimate

$$\|x^*(\tau_1^*)-x(\tau_1)\| \leq \|x^*(\tau_1^*)-x^*(\tau_1)\|+\|x^*(\tau_1)-x(\tau_1)\|<M_1\eta+\varepsilon$$

The last estimate enables us to write

$$\lim_{\mu^*\to\mu(\mu^*\in M)} x^*(\tau_1^*)=x(\tau_1)$$

whence, in view of condition (D), we conclude that there exists $\delta^{iv}>0$ such that if $\mu^*\in M$ and $|\mu^*-\mu|<\delta^{iv}$, then

$$\|I_1(x^*(\mu^*), \mu^*)-I_1(x(\mu), \mu)\|<\varepsilon$$

Then

$$\|x_1^+-x^*(\tau_1^*)-I_1(x^*(\tau_1^*), \mu^*)\|<2\varepsilon+M_1\eta$$

which also implies that

$$\lim_{\mu^*\to\mu(\mu^*\in M)} [x^*(\tau_1^*)+I_1(x^*(\tau_1^*), \mu^*)]=x_1^+ \tag{26}$$

Consider the sets $A=\{x; x=x(t; \tau_1, x_1^+, \mu), \tau_1\leq t\leq T\}$ and $A_1=\{x; \alpha_1(x)=0, x\in D\}$. They are closed and A is bounded. Then

$$\rho(A, A_1)=\beta>0$$

Consider also the functions $x(t; \tau_1, x_1^+, \mu)$ and $x(t; \tau_1^*, x^*(\tau_1)+I_1(x(\tau_1^*), \mu^*), \mu^*)$. Respectively for $t>\tau_1$ and $t>\tau_1^*$ they coincide with the solutions $x(t)$ of problem (4)-(6) and $x^*(t)$ of problem (15)-(17). From relations (24), (26), and the theorem of continuous dependence it follows that there exists $\delta^v>0$ such that if $\mu^*\in M$ and $|\mu^*-\mu|<\delta^v$, then

$$\|x(t; \tau_1, x_1^+, \mu)-x(t; \tau_1^*, x^*(\tau_1^*)+I_1(x(\tau_1^*), \mu^*), \mu^*)\|$$
$$<\min(\varepsilon, \beta), \qquad \tau_1^*<t\leq T$$

The last inequality also implies that for $\tau_1^*<t\leq T$ the solution $x^*(t)$ does not nullify the function α_2 and the following inequality holds:

$$\|x^*(t)-x(t)\|<\varepsilon, \qquad \tau_1^*<t\leq T \tag{27}$$

The assertion of the theorem follows from (25), (27), and the estimate $|\tau_1^*-\tau_1|<\eta$.

This completes the proof of Theorem 2. ■

Theorem 3. Let conditions (A)-(D) hold. Then the solution of problem (4)-(6) depends continuously on the parameter $\mu\in M$ for $\tau_0\leq t\leq T$.

Proof. From Theorem 1 it follows that in the interval $[\tau_0, T]$ a finite number of impulse moments are contained, i.e., there exists a number k such that $\tau_k\leq T<\tau_{k+1}$.

If $k=0$, i.e., if the inequality $\tau_1 > T$ holds, then Theorem 3 follows from Lemma 2 and the theorem of continuous dependence.

Let $k \geq 1$. Introduce the notations $t_0 = \tau_0$, $t_i = (\tau_i + \tau_{i+1})/2$, $i = 1, 2, \ldots, k-1$, $t_k = T$. From Theorem 2 we obtain that the solution $x(t)$ of problem (4)-(6) depends continuously on the parameter μ for $t_0 \leq t \leq t_1$. Since

$$\lim_{\mu^* \to \mu (\mu^* \in M)} x^*(t_1) = x(t_1)$$

again by Theorem 2 we obtain that the solution $x(t)$ depends continuously on the parameter μ for $t_1 \leq t \leq t_2$. The proof of the theorem is completed by induction with respect to the number of the intervals $[t_{i-1}, t_i]$, $i = 1, 2, \ldots, k$.

Thus, Theorem 3 is proved. ■

Corollary 1. Let conditions (A), (B), and (D) hold.

Then the solution of problem (4), (5) with initial condition $x(\tau_0, \mu) = x_0$, where $x_0 \in D$, depends continuously on the parameter $\mu \in M$ for $\tau_0 \leq t \leq T$.

Remark 1. The results obtained up to now still hold if the parameter $\mu \in R^m$, i.e., $\mu = (\mu_1, \ldots, \mu_m)$. In this case the proofs of the assertions considered are carried out separately for each of the components of the parameter.

Consider the following initial value problem (without a parameter):

$$dy/dt = g_i(t, y), \qquad \theta_{i-1} < t \leq \theta_i \tag{28}$$

$$\Delta y(t)|_{t=\theta_i} = J_i(y(\theta_i)), \qquad i = 1, 2, \ldots \tag{29}$$

$$y(\theta_0) = y_0 \tag{30}$$

where $g_i : R^+ \times D \to R^n$, $J_i : D \to R^n$, $\theta_0 \geq 0$, and $y_0 \in D$. The moments $\theta_1, \theta_2, \ldots$ are determined analogously to $\tau_1, \tau_2, \ldots$. Henceforth, by $y(t; \theta_0^*, y_0^*)$ we shall denote the solution of problem (28), (29) with initial condition

$$y(\theta_0^*) = y_0^*, \qquad \theta_0^* \geq 0, \qquad y_0^* \in D$$

Definition 2. We shall say that the solution of problem (28)-(30) depends continuously on the initial condition for $\theta_0 \leq t \leq T$ if

$$(\forall \varepsilon, \eta > 0)(\exists \delta = \delta(\varepsilon, \eta) > 0)(\forall y_0^* \in D, \|y_0^* - y_0\| < \delta)(\forall \theta_0 \geq 0) \Rightarrow$$

$$\|y(t; \theta_0, y_0^*) - y(t; \theta_0, y_0)\| < \varepsilon, \qquad t \geq \theta_0, \quad |t - \theta_i| > \eta, \qquad i = 1, 2, \ldots$$

Corollary 2. For any $i = 1, 2, \ldots$ let the following conditions hold:

1. The functions g_i are Lipschitz continuous in $R^+ \times D$ with constant F.

2. There exist constants $M_i > 0$ such that $\|g_i(t, y)\| \le M_i$, $(t, y) \in R^+ \times D$.

3. For any point $(\theta_0^*, y_0^*) \in R^+ \times D$ the solution of the problem

$$dy/dt = g_i(t, y), \qquad y(\theta_0^*) = y_0^*$$

does not leave the set D.

4. There exist constants $\rho_i > 0$ for which one of the following two conditions is fulfilled:

4.1. (i) $(I + J_i)$: $A_i \to D$; (ii) $|\alpha_i(y + J_{i-1}(y))| \ge \rho_i$, $y \in A_{i-1}$, $J_0(y) = 0$.

4.2. (i) $(I + J_{i-1})$: $A_{i-1} \to A_i$; (ii) there exists at least one point θ, $\theta_{i-1} < \theta < \theta_i$, such that

$$|\langle \partial\alpha_i(y(\theta; \theta_0, y_0))/\partial y, g_i(\theta, y(\theta_i; \theta_0, y_0))\rangle| \ge \rho_i$$

5. Conditions A4, A5, and A7 hold.

6. There exist positive constants δ_i such that for any two points θ', θ'', $\theta_i - \delta_i < \theta' < \theta_i$ and $\theta_i < \theta'' < \theta_i + \delta_i$, the following inequality holds:

$$\alpha_i(y_i(\theta'))\langle \partial\alpha_i(y_i(\theta''))/\partial y, g_i(\theta'', y_i(\theta''))\rangle < 0$$

where by $y_i(t)$ we have denoted respectively the solutions of the problems

$$dy/dt = g_i(t, y), \qquad y_i(\theta_{i-1}) = y_{i-1}(\theta_{i-1}) + J_{i-1}(y_{i-1}(\theta_{i-1}))$$

7. The functions J_i are continuous in D.

Then the solution of problem (28)-(30) depends continuously on the initial condition for $\theta_0 \le t \le T$, $\theta_0 \ge 0$.

Proof. By the substitution $z = y - y_0$, problem (28)-(30) is transformed into the problem

$$\partial z/\partial t = g_i(t, z + y_0), \qquad \theta_{i-1} < t \le \theta_i \tag{31}$$

$$\Delta z(t)|_{t=\theta_i} = J_i(z(\theta_i) + y_0), \qquad i = 1, 2, \ldots \tag{32}$$

$$z(\theta_0) = 0 \tag{33}$$

The last problem satisfies the conditions of Theorem 3 with a parameter $\mu = y_0$, whence, in view of Remark 1, we obtain that the solution of problem (31)-(33) depends continuously on the parameter y_0 for $\theta_0 \le t \le T$. Hence, the solution $y(t; \theta_0, y_0)$ depends continuously on the initial condition for $\theta_0 \le t \le T$.

Thus, Corollary 2 is proved. ■

5. APPLICATION

We shall consider the system of differential equations simulating the action of a hydraulic valve (Figure 1). Assume that at the initial moment

($\tau_0 = 0$) the valve is in state 2, i.e., the simulating system has the initial condition

$$(x, y, P_1)|_{t=0} = (0, 0, P_{10})$$

where $\varphi(P_{10}) < 0$. Hence,

$$\pi D^2(P_{10} - P_2)/4 - c_0 < 0$$

According to (1) and (2), the system has the form

$$\dot{x} = 0 \tag{34}$$

$$\dot{y} = 0 \tag{35}$$

$$\dot{P}_1 = EQ_1/V \tag{36}$$

for $\tau_{2i-2} < t \leq \tau_{2i-1}$ and

$$\dot{x} = y \tag{37}$$

$$\dot{y} = [\pi D^2(P_1 - P_2)/4 - c_0 - c_s x]/m \tag{38}$$

$$\dot{P}_1 = E\{Q_1 - \nu\pi Dx \sin\alpha\, [2(P_1 - P_2)/\omega]^{1/2}\}/V \tag{39}$$

for $\tau_{2i-1} < t \leq \tau_{2i}$, i.e., the following equalities hold:

$$\begin{aligned} &f_{2i-1}(t, x, y, P_1) = (0, 0, EQ_1/V) \\ &f_{2i}(t, x, y, P_1) \\ &\quad = (y, [\pi D^2(P_1 - P_2)/4 - c_0 - c_s x]/m; \\ &\quad\quad E\{Q_1 - \nu\pi Dx \sin\alpha\, [2(P_1 - P_2)/\omega]^{1/2}\}/V) \end{aligned} \tag{40}$$

The impulse perturbations are given by means of the equalities

$$\Delta(x, y, P_1)|_{t=\tau_{2i-1}} = 0, \qquad \Delta(x, y, P_1)|_{t=2i} = -y(\tau_i), \qquad i = 1, 2, \ldots$$

The functions by means of which the impulse moments $\tau_1, \tau_2, \ldots$ are determined have the form

$$\alpha_{2i-1} = \pi D^2(P_1 - P_2)/4 - c_0, \qquad \alpha_{2i} = x, \qquad i = 1, 2, \ldots$$

Further we assume that:

(i) The definition domains of the functions (40) are bounded, i.e., $0 = x_* \leq x \leq x^*$, $y_* \leq y \leq y^*$, $P_2 \leq P_* \leq P_1 \leq P^*$, where x^*, y_*, y^*, P_*, and P^* are real constants.

(ii) The function $Q_1 = Q_1(t)$ is constant.

(iii) Each of the parameters is bounded.

We shall show that the simulating system with variable structure and impulses satisfies conditions (A)-(D). Indeed, since the right-hand sides of equations (34)-(38) do not depend on t and are linear with respect to the variables x, y, and P_1, then it follows that they are Lipschitz continuous in their definition domain with constants independent of the parameters. We shall show that the right-hand side of (39) is also Lipschitz continuous. Let (x', P_1') and (x'', P_1'') be two points of the definition domain of this function. Then

$$\begin{aligned}
&|E\{Q_1 - \nu\pi D x'' \sin\alpha\,[2(P_1'' - P_2)/\omega]^{1/2}\}/V \\
&\quad - E\{Q_1 - \nu\pi D x' \sin\alpha\,[2(P_1' - P_2)/\omega]^{1/2}\}/V| \\
&\quad \le K(|x'' - x'|(P_1' - P_2)^{1/2} + x'|(P_1'' - P_2)^{1/2} - (P_1' - P_2)^{1/2}|) \\
&\quad \le K(P_1^* - P_2)^{1/2}|x'' - x'| + [Kx^*/2(P_* - P_2)^{1/2}]|P_1'' - P_1'|
\end{aligned}$$

where $K = E\nu\pi D \sin\alpha\,(2/\omega)^{1/2}/V$. Conditions A2–A5 are also satisfied by the system considered. We shall verify condition A6. Taking into account

$$I_{2i-1} = 0, \qquad I_{2i} = -y, \qquad i = 1, 2, \ldots$$

we immediately see the validity of condition A6.1i. Condition A6.2i holds for even indices. Indeed, from the analytical expressions for the functions α_{2i-1} and α_{2i}, we establish that

$$\begin{aligned}
A_{2i-1} &= \{(x, y, P_1);\, 0 \le x \le x^*,\, y_* \le y \le y^*,\, P_1 = \kappa\}, \qquad \kappa = 4c_0/\pi D^2 + P_2 \\
A_{2i} &= \{(0, y, P_1);\, y_* \le y \le y^*,\, P_* \le P_1 \le P^*\}
\end{aligned}$$

From the fact that the functions α_{2i-1} vanish when the valve is in a closed state, it follows that $x = 0$ and $y = 0$, i.e., we can assume that

$$A_{2i-1} = \{(0, 0, \kappa)\}$$

Then we obtain

$$I + I_{2i-1} = I\colon \quad A_{2i-1} \to A_{2i-1} \subset A_{2i}$$

which we wanted to show. The existence of nonnegative constants ρ_i satisfying conditions A6.1ii and A6.2ii is obvious. Thus, for instance, for even indices, condition A6.2ii is satisfied for the following choice of the constants ρ_{2i}. For this purpose we first note that

$$\partial\alpha_{2i}/\partial(x, y, P_1) = (1, 0, 0), \qquad i = 1, 2, \ldots$$

whence we obtain

$$\langle \alpha_{2i}/\partial(x, y, P_1), f_{2i}\rangle = y(t), \qquad \tau_{2i-1} < t \le \tau_{2i}$$

Hence, in this case we can set

$$\rho_{2i} = \max_{\tau_{2i-1} \le t \le \tau_{2i}} |y(t)|$$

Taking into account that the right-hand side of the system of the mathematical model changes alternately, we establish that the series of condition A7 and the series $\rho_1 + \rho_2 + \cdots$ are both convergent or both divergent.

Conditions (B) and (D) are verified trivially. Condition (C) is also satisfied. In fact, since

$$\partial \alpha_{2i-1}/\partial(x, y, P_1) = (0, 0, \pi D^2/4)$$

we obtain

$$\begin{aligned}
&\alpha_{2i-1}(x_{2i-1}(\tau'), y_{2i-1}(\tau'), P_{1(2i-1)}(\tau')) \\
&\qquad \times \langle \partial \alpha_{2i-1}(x_{2i-1}(\tau''), y_{2i-1}(\tau''), P_{1(2i-1)}(\tau''))/\partial(x, y, P_1) \\
&\qquad f_{2i-1}(\tau'', x_{2i-1}(\tau''), y_{2i-1}(\tau''), P_{1(2i-1)}(\tau''), w)\rangle \\
&= [\pi D^2(P_{1(2i-1)}(\tau') - \dot{P}_2)/4 - c_0]\pi D^2 EQ/4V
\end{aligned}$$

where $(x_{2i-1}, y_{2i-1}, P_{1(2i-1)})$ are solutions of the system (34)-(36) with initial condition

$$(x_{2i-1}(t), y_{2i-1}(t), P_{1(2i-1)}(t))|_{t=\tau_{2i-2}} = (0, 0, P_{1(2i-2)}(\tau_{2i-2}))$$

The points $\tau' \in (\tau_{2i-1} - \delta_{2i-1}, \tau_{2i-1})$ and $\tau'' \in (\tau_{2i-1}, \tau_{2i-1} + \delta_{2i-1})$, where the constants δ_{2i-1} are small enough. For $\delta_{2i-1} < \tau_{2i-1} - \tau_{2i-2}$ the inequality

$$\pi D^2(P_1(\tau') - P_2)/4 - c_0 < 0, \qquad \tau_{2i-2} < \tau' < \tau_{2i-1}$$

holds, which implies that the inequality in condition (C) is satisfied for odd indices. For even indices we have

$$\begin{aligned}
&\alpha_{2i}(x_{2i}(\tau'), y_{2i}(\tau'), P_{1(2i)}(\tau')) \\
&\qquad \times \langle \partial \alpha_{2i}(x_{2i}(\tau''), y_{2i}(\tau''), P_{1(2i)}(\tau''))/\partial(x, y, P_1) \\
&\qquad f_{2i}(\tau'', x_{2i}(\tau''), y_{2i}(\tau''), P_{1(2i)}(\tau''), w)\rangle \\
&= x_{2i}(\tau') \cdot y_{2i}(\tau'')
\end{aligned} \tag{41}$$

where $(x_{2i}, y_{2i}, P_{1(2i)})$ are solutions of the system (37)-(39) with initial condition

$$(x_{2i}(t), y_{2i}(t), P_{1(2i)}(t))|_{t=\tau_{2i-1}} = (0, 0, \kappa)$$

In equality (41) the points $\tau' \in (\tau_{2i} - \delta_{2i}, \tau_{2i})$ and $\tau'' \in (\tau_{2i}, \tau_{2i} + \delta_{2i})$, where the constants δ_{2i} are small enough. The following inequalities hold:

$$x_{2i}(\tau') > 0, \qquad \tau_{2i-1} < \tau' < \tau_{2i}$$

$$y_{2i}(\tau_{2i} + 0) < 0$$

The last inequality implies that if the constant δ_{2i} is small enough, then $y_{2i}(\tau'') < 0$ for $\tau_{2i} < \tau'' < \tau_{2i} + \delta_{2i}$. But then

$$x_{2i}(\tau')y_{2i}(\tau'') < 0, \qquad \tau_{2i} - \delta_{2i} < \tau' < \tau_{2i}, \qquad \tau_{2i} < \tau'' < \tau_{2i} + \delta_{2i}$$

whence by means of (41) we conclude that condition (C) holds in this case, too.

Hence, if the series $\rho_1 + \rho_2 + \cdots$ is divergent, then the solution of the problem of Example 1 depends continuously on the initial condition and the parameters.

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